

Optimality of Non-Restarting CUSUM charts

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Abstract

We show optimality, in a well-defined sense, using cumulative sum (CUSUM) charts for detecting changes in distributions. We consider a setting with multiple changes between two known distributions. This result advocates the use of non-restarting CUSUM charts with an upper boundary. Typically, after signalling, a CUSUM chart is restarted by setting it to some value below the threshold. A non-restarting CUSUM chart is not reset after signalling; thus is able to signal continuously. Imposing an upper boundary prevents the CUSUM chart rising too high, which facilitates detection in our setting. We discuss, via simulations, how the choice of the upper boundary changes the signals made by the non-restarting CUSUM charts.

Key words: CUSUM chart, optimality, non-restarting, upper boundary, continuous signals

1 Introduction

In statistical process control, one of the most commonly used control charts is the cumulative sum (CUSUM) chart developed by Page (1954). As a simple example of a CUSUM chart consider the following. Assume we sequentially observe the independent random variables X_t ($t \in \mathbb{N} = \{1, 2, \dots\}$). Suppose the observations X_1, \dots, X_{n-1} each have distribution $N(0, 1)$ and X_n, X_{n+1}, \dots have $N(\Delta, 1)$ for some known $\Delta > 0$. The aim is then to find the unknown change point $n \in \mathbb{N}$. The classic CUSUM chart is

$$S_t = \max\{S_{t-1} + X_t - \Delta/2, 0\}, \quad S_0 = 0.$$

The chart signals a change at time $\inf\{t > 0; S_t \geq \alpha\}$ for some threshold $\alpha > 0$. At this time the chart suspects a change in distribution. When the CUSUM chart crosses the threshold α , it is restarted in some fashion. The chart is typically reset at 0, with some practitioners opting for the headstart feature (Lucas and Crosier, 1982), where the chart is reset at a different value such as $\alpha/2$. CUSUM charts were initially designed for use in industry (Page, 1954), where restarting coincides with a machine being repaired and reset.

We are concerned with situations where the observations can switch multiple times between two known distributions and where resetting is not possible. For example, medical settings where “machines” such as hospitals cannot be reset when a deterioration of performance is suspected. Thus we shall use the CUSUM charts of the form

$$R_t = f_t(R_{t-1}) \quad \text{where} \quad f_t(x) = \min\{\max(x + \log \ell(X_t), 0), h\} \quad (1)$$

for $t \in \mathbb{N}$, where $\ell(x)$ is the Radon-Nikodym derivative of F_1 with respect to F_0 and $h > 0$ is a constant specifying an upper boundary. As explained in Gandy and Lau (2012), a single

non-restarting CUSUM chart (1) with $R_0 = 0$ is appropriate in settings where restarting is not possible. Moreover, it is shown that the imposition of an upper boundary facilitates the detection when switching between an in-control state and out-of-control state many times. The CUSUM charts defined by (1) constitutes a family of charts, each member represented by a specific choice of starting value $R_0 \in [0, h]$. We distinguish two particular CUSUM charts which we shall use throughout this paper.

Definition 1 *For a specified upper boundary $h > 0$, denote the CUSUM chart of the form (1) with $R_0 = 0$ as R_t^L and with $R_0 = h$ as R_t^U .*

These are the “extreme” charts in (1) i.e. for any $R_0 \in [0, h]$ we have $R_t^L \leq R_t \leq R_t^U$ for all $t \in \mathbb{N}$.

In this paper, it is shown that using CUSUM charts of the form (1) are optimal in a well-defined sense. The optimality criteria we consider is similar to that used by Lorden (1971). Our optimality is a generalisation of the Moustakides (1986) result, where a single change in distribution is considered. Showing optimality of CUSUM charts is a subject of interest in the change point detection literature. For instance, Poor (1998) extends the optimality in the sense of Lorden (1971), to include a exponential penalty for delay. Moreover, extensions to continuous time processes (Moustakides, 2004), use of dependent observations such as Markov chains (Yakir, 1994) and random processes (Moustakides, 1998) have also been explored. However, to our knowledge, the optimality of CUSUM charts in a setting where observations can switch between two known distributions multiple times has not been investigated.

To make clear our setting and the charts we shall be using, consider the following illustrative example. Let $X_t \sim N(-1/2, 1)$ when in the in-control state and $X_t \sim N(1/2, 1)$ when in the out-of-control state at time t . In this example, we use the charts R_t^L and R_t^U with $h = 16$ with threshold $\alpha = 8$. These non-restarting CUSUM charts will signal out-of-control at time t if $R_t^L \geq \alpha$. Moreover, these charts will continually signal out-of-control whilst R_t^L remains above α . Similarly, for signalling in-control when the upper chart R_t^U drops below α .

In Figure 1 it is clear that the lower boundary at 0 and upper boundary at h are “holding barriers” that prevent the chart dropping too low or rising too high. Beneath the plot of the CUSUM chart in Figure 1, is the signal indicator. The grey areas represent the charts signalling out-of-control and in-control, where it is clear that continuous signals are made. This would not be the case when using a traditional CUSUM chart (i.e. a chart of the form (1) with $h = \infty$ starting at 0) that restarts when its threshold is crossed.

In Section 2.1 we begin by introducing notation appropriate in our setting. We then introduce stochastic processes as signal processes, that allows continuous signalling. In Section 2.2 we present Lorden (1971) criterion of optimality as used in Moustakides (1986). Section 2.3 contains the main theoretical result of the paper. Here, it is shown that using non-restarting CUSUM charts with an upper boundary are optimal in a well-defined sense. Although the upper boundary is introduced in the theoretical result, its value remains unspecified. In Section 3 we explore how the choice of the upper boundary affects the signals made and practical consequences. We conclude in Section 4 we some remarks concerning future topics of research.

2 Optimality

In this section we present our optimality criteria. We present the theoretical result proving that using non-restarting CUSUM charts with an upper boundary, are optimal in the sense defined. We begin by introducing notation and recalling the Moustakides (1986) result.

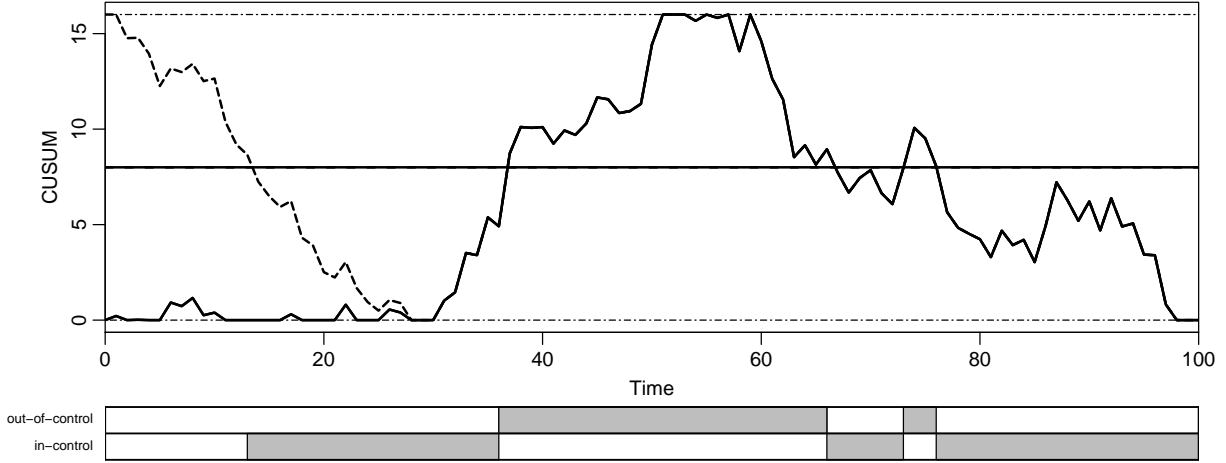


Figure 1: Two non-restarting CUSUM charts (above) with signal indicator (below).

2.1 Notation

Let (Ω, \mathcal{F}) be a measurable space with random variables $(X_t : t \in \mathbb{N})$. Let \mathcal{P} be a set of probability measures on (Ω, \mathcal{F}) such that for all $P \in \mathcal{P}$, X_t ($t \in \mathbb{N}$) are independent and the distribution of X_t , $\mathcal{L}(X_t) \in \{F_0, F_1\}$ for all $t \in \mathbb{N}$. Both F_0 and F_1 are assumed to be known and mutually absolutely continuous. We will assume that $\ell(X_1)$ has no atoms with respect to $P \in \mathcal{P}$ such that $\mathcal{L}(X_t) = F_0$ for all $t \in \mathbb{N}$. The σ -algebra generated by $\{X_1, \dots, X_t\}$ is denoted by \mathcal{F}_t . Henceforth, whenever we use an expectation E or essential supremum, we shall specify with respect to which probability measure in \mathcal{P} by defining the distributions of the X_t ($t \in \mathbb{N}$). We shall refer to the case $\mathcal{L}(X_t) = F_0$ as X_t being in the in-control state and $\mathcal{L}(X_t) = F_1$ as X_t being in the out-of-control state.

As we are concerned with detecting periods where the observations are in the in-control state or the out-of-control state, it is not convenient to work with stopping times as used in Moustakides (1986) and Lorden (1971). Thus we shall use a stochastic process as a signal process.

Definition 2 A signal process is a stochastic process $Z = \{Z_t : t \in \mathbb{N}\}$ such that $Z_t \in \{0, 1, \emptyset\}$ and Z_t is \mathcal{F}_t -measurable for all $t \in \mathbb{N}$.

For a signal process Z , the events $\{Z_t = j\}$ correspond to signalling in-control ($j = 0$), signalling out-of-control ($j = 1$) and no signal ($j = \emptyset$) at time t . The first in-control ($j = 0$) or out-of-control ($j = 1$) signal by the signal process Z after time n is denoted by $\tau_n^j(Z) = \inf\{t \geq n : Z_t = j\}$.

Lastly, we define CUSUM chart developed by Page (1954) S_t^L and its analogue, with the roles of F_0 and F_1 swapped S_t^U as follows.

Definition 3 For $h = \infty$ and f_t as in (1) let, for all $t \in \mathbb{N}$

$$S_t^L = f_t(S_{t-1}^L) \quad \text{and} \quad S_t^U = -f_t(-S_{t-1}^U) \quad \text{with} \quad S_0^L = S_0^U = 0.$$

2.2 Moustakides (1986) Optimality Result

In this section we present the optimality result of Moustakides (1986) in terms of signal processes. First, we recall optimality criteria used in Lorden (1971).

Definition 4 Let $P \in \mathcal{P}$ be such that $\mathcal{L}(X_t) = F_{1-j}$ for all $t < n$ and $\mathcal{L}(X_t) = F_j$ for all $t \geq n$. For a signal process Z and $j \in \{0, 1\}$ let

$$\begin{aligned} \mathbf{D}_n^j(Z) &= \text{esssup } E \left\{ \left[\tau_1^j(Z) - n + 1 \right]^+ \mid \mathcal{F}_{n-1} \right\}, \\ \mathbf{D}^j(Z) &= \sup_{n \in \mathbb{N}} \mathbf{D}_n^j(Z). \end{aligned}$$

The term $\mathbf{D}^j(Z)$ is the longest average delay of signalling the X 's are F_j distributed, guaranteed regardless of the behaviour of the F_{1-j} distributed X 's before the change.

For a constant $\gamma_1 > 0$, consider the optimization problem posed in Moustakides (1986)

$$\begin{cases} \mathbf{D}^1(Z) \rightarrow \min \\ E\{\tau_1^1(Z) \mid \mathcal{L}(X_t) = F_1 \forall t \in \mathbb{N}\} \geq \gamma_1 \end{cases} \quad (2)$$

where Z is a signal process. As proved in Ritov (1990, Proposition 2) a solution of (2) is the signal process Z^1 where $Z_t^1 = 1$ if $S_t^L \geq k_L$ for $t \in \mathbb{N}$, where $k_L > 0$ is a constant determined by γ_1 . We appeal to this alternative proof of Moustakides (1986) to avoid complications with the σ -algebras. By swapping the roles of F_0 and F_1 we obtain an analogous result. More precisely, for a constant $\gamma_0 > 0$, consider the optimization problem

$$\begin{cases} \mathbf{D}^0(Z) \rightarrow \min \\ E\{\tau_1^0(Z) \mid \mathcal{L}(X_t) = F_0 \forall t \in \mathbb{N}\} \geq \gamma_0 \end{cases} \quad (3)$$

where Z is a signal process. A solution of (3) is the signal process Z^0 where $Z_t^0 = 0$ if $S_t^U \geq k_U$ for $t \in \mathbb{N}$, where $k_U > 0$ is a constant determined by γ_0 .

The expectation in (2) is average time until signalling out-of-control, when all the observations are truly in-control and vice versa for the expectation in (3). This is referred to as the average in-control and out-of-control run lengths respectively.

2.3 Main Result

In this section we present our optimality result. We begin by defining our optimality criteria.

Definition 5 For a signal process Z and $j \in \{0, 1\}$ let

$$\begin{aligned} \mathbf{C}_n^j(Z) &= \max \left\{ \text{esssup } E \left(\tau_n^j(Z) - n + 1 \mid \mathcal{F}_{n-1} \right) : P \in \mathcal{P}, \mathcal{L}(X_t) = F_j \forall t \geq n \right\}, \\ \mathbf{C}^j(Z) &= \sup_{n \in \mathbb{N}} \mathbf{C}_n^j(Z). \end{aligned}$$

For constants $c_1 > 0$ and $c_2 > 0$, consider the optimization problem

$$\begin{cases} \mathbf{C}^1(Z) \rightarrow \min \\ \mathbf{C}^0(Z) \rightarrow \min \\ B^0(Z) \geq c_0 \\ B^1(Z) \geq c_1 \end{cases} \quad (4)$$

where Z is a signal process and for $j \in \{0, 1\}$ with $P \in \mathcal{P}$ such that $\mathcal{L}(X_t) = F_j$ for all $t \in \mathbb{N}$

$$B^j(Z) = \inf_{n \in \mathbb{N}} B_n^j(Z) \text{ where } B_n^j(Z) = \text{esssup } E \{ \tau_n^j(Z) - n + 1 \mid \mathcal{F}_{n-1} \}.$$

The term $\mathbf{C}^1(Z)$ ($\mathbf{C}^0(Z)$) is the longest average delay of signalling out-of-control (in-control), guaranteed regardless of the distribution of the X 's prior to the permanent change out-of-control (in-control) and prior signals made. Unlike $\mathbf{D}^j(Z)$ which considers the first stopping time $\tau_1^j(Z)$, we consider the reset stopping time $\tau_n^j(Z)$. Hence it is possible to repeatedly signal in-control or out-of-control. Each $B_n^j(Z)$, $n \in \mathbb{N}$, is the in-control ($j = 1$) or out-of-control ($j = 0$) average run length after the change point n .

Theorem 1 *There exists constants $k_U > 0$ and $k_L > 0$ such that for any $h \geq \max(k_U, k_L)$ a solution of (4) is the signal process Z^* defined, for $t \in \mathbb{N}$, by*

$$Z_t^* = \begin{cases} 0, & R_t^U \leq h - k_U \\ 1, & R_t^L \geq k_L \\ \emptyset, & R_t^L < k_L, R_t^U > h - k_U \end{cases}.$$

The proof of Theorem 1 is in Appendix 1.

Theorem 1 uses the charts R_t^L and R_t^U , which both follow the same evolutionary equation f_t . However, R_t^L starts at value 0 whereas R_t^U starts at the upper boundary h . Starting at these two values means that we consider both the cases where the observation start (at time 0) in the in-control state and the out-of-control state simultaneously. Thus, the prejudice of assuming the observations are initially in the in-control state or the out-of-control state is removed. Further, Theorem 1 introduces the upper boundary h that prevents the CUSUM chart rising too high. Gandy and Lau (2012) explain how using an upper boundary in non-restarting CUSUM charts helps detect periods of out-of-control and in-control activity. Although the upper boundary h has been introduced in Theorem 1, its value is not specified. This practical issue is discussed in detail in Section 3 where we explore how varying the upper boundary affects the signal process.

3 Upper Boundary

In this section we investigate how varying the upper boundary h affects the number of false and correct signals made by the charts. We begin by classifying different scenarios that can occur by changing h . An illustrative example demonstrates how these scenarios leads to different signals for the same data.

3.1 Signal Gap and Overlay

Setting $h = k_U + k_L$ in Theorem 1 means that R_t^L and R_t^U signal whenever k_L is crossed. We shall refer to this as the single threshold case. However, this need not be the case as Theorem 1 just stipulates that $h \geq \max(k_U, k_L)$. When $h > k_U + k_L$ then it is possible to simultaneously signal in-control and out-of-control. We shall refer to this case as a signal overlay. When $h < k_U + k_L$, then no signal can occur when either chart is between $h - k_U$ and k_L . We shall refer to this as a signal gap. Thus both a gap and overlay are intervals where the charts cannot signal in-control or out-of-control with confidence. If we consider a simultaneous signal as no signal due to ambiguity, the cases of signal gap and overlay are the same. Thus, henceforth we shall only consider $h \leq k_U + k_L$.

To see the disparity between different values of the upper boundary h consider the following example. Let $X_t \sim N(-1/2, 1)$ when in the in-control state and $X_t \sim N(1/2, 1)$ when in the out-of-control state at time t . In this example we set the thresholds at $k_U = k_L = 5$. In practice, however, it is typical to set these thresholds by pre-specifying the average run length. This can be achieved by running simulations of the CUSUM charts and conducting a numerical search. Alternatively, one could approximate the distribution of the CUSUM chart by a discrete Markov chain and obtain the average run length using the transition matrix (Brook and Evans, 1972). Using the Brook and Evans (1972) approach with 100 states, the average in-control and out-of-control run length with $k_U = k_L = 5$ is approximately 930 time units. In this illustrative example we choose the out-of-control periods as time 16 to 35 and time 51 to 60. In Figure 2 we plot the two CUSUM charts R_t^L and R_t^U based on the same random realization of X_1, X_2, \dots for $h = 6, 8$ and 10 .

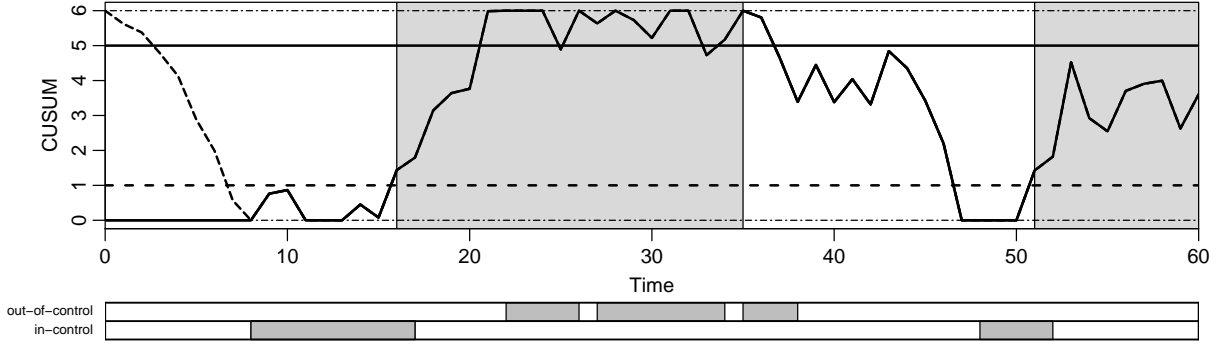
Figure 2 illustrates some important points. First choosing a different value of h leads to a different signalling process for the same data. This is clearly presented by the signal indicators under Figure 2a, 2b and 2c. Second, once the two CUSUM charts have coupled into a single chart in the single threshold case ($h = k_U + k_L$), definitive signals are given. More precisely, in Figure 2c, after time 13, the chart signals either in-control or out-of-control. This is not the case where a signal gap occurs. Comparing Figure 2a and 2b, we see that as the signal gap increases, fewer incorrect signals are made at the expense of fewer overall signals. This point is explored further in Section 3.2. Lastly, in this example, the two CUSUM charts eventually coalesce or couple into a single chart. The time until coupling depends, not only on F_0 and F_1 , but also on the upper boundary h . Coupling occurs at value 0 or h by construction of the CUSUM charts. Moreover, if we were the run CUSUM charts of the form (1) starting from every $R_0 \in [0, h]$ all charts will couple either at 0 or h . This follows from the fact that R_t^L and R_t^U represent the “extreme” CUSUM charts. Thus at the time of coupling we signal confidently in-control, if at 0, or out-of-control, if at h , since all possible charts of the form (1) agree.

3.2 Practical Implications

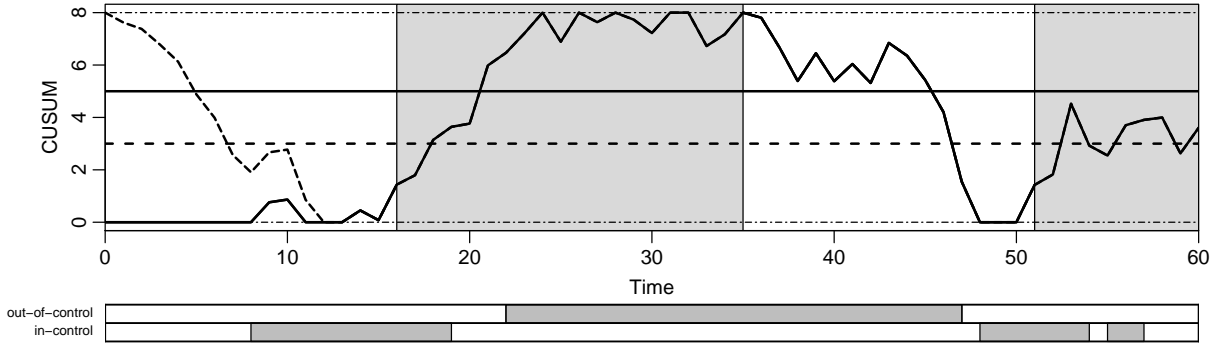
Although we have discussed the scenarios that arise with different choices of the upper boundary, we have yet to explore the practical implications. As illustrated in Section 3.1, the choice of h affects the signals made by the charts. We now explore how the choice of h affects the number of correct and false signals in a small simulation.

We use the same in-control and out-of-control distributions and thresholds used Section 3.1. The true in-control and out-of-control periods are represented by the grey areas in Figure 3. For a single iteration, we use the CUSUM charts R_t^L and R_t^U with $h = 6, 8, 10$ all with the same seed. We repeated this 10,000 times. Figure 3a and 3b are plots the average number of false signals and correct signals, respectively, pointwise in time.

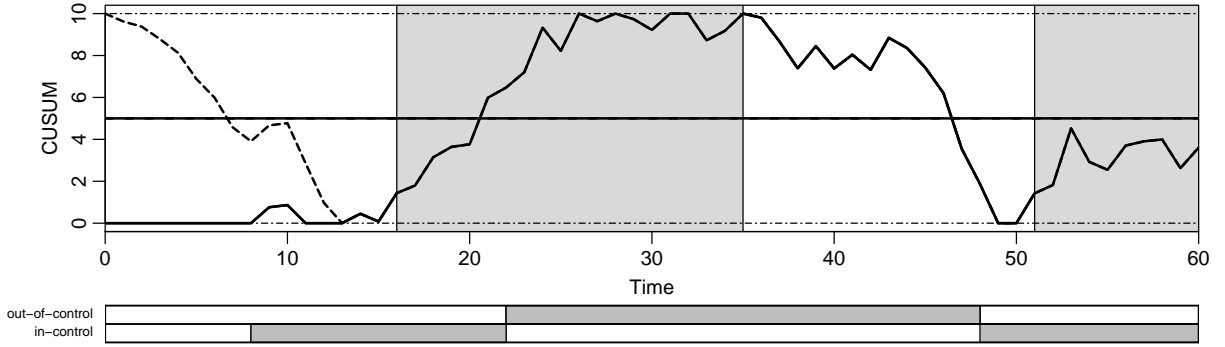
In Figure 3a, when switching between states, there is a sudden peak in the number of false signals which subsequently declines. The single threshold case where $h = 10$, leads to the an overall higher false and correct average than the other values of h . From Figure 3 the notion of a signal gap can be interpreted as follows: Whilst a chart is in a signal gap, signalling is deferred until a more “definitive” signal can be made i.e. when the chart exits the gap. This results in fewer false signals being made, with the trade-off that fewer signals overall are made. In practice, this could represent the users attitude to “err on the side of caution” and patiently waiting to signal with more confidence.



(a) $h = 6$ Signal Gap

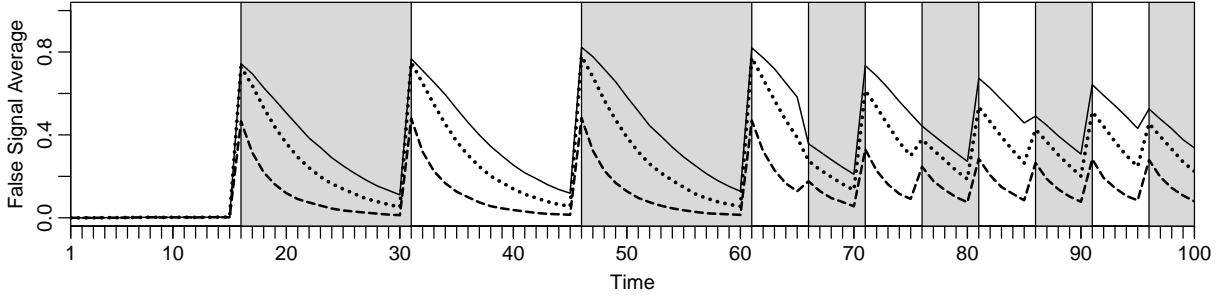


(b) $h = 8$ Signal Gap

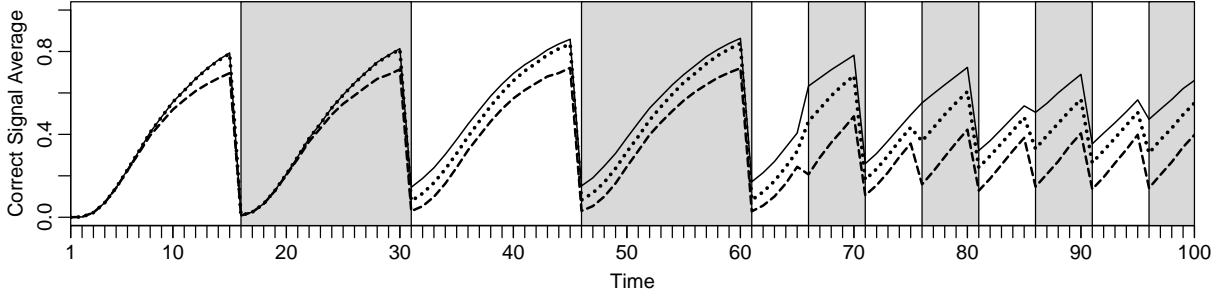


(c) $h = 10$ Single Threshold

Figure 2: Upper figures contain one lower (solid) and upper (dashed) chart with lower (solid) and upper (dashed) threshold. Grey area represents the out-of-control period. Underneath is the signal indicator with grey areas representing signals.



(a) Average Number of False Signals Pointwise in Time



(b) Average Number of Correct Signals Pointwise in Time

Figure 3: Average number of false and correct signals for $h = 10$ (solid): same threshold case, $h = 8$ (dotted) and $h = 6$ (dashed): both signal gap cases.

4 Discussion

We have shown that using non-restarting CUSUM charts of the form (1) are optimal in the sense of (4). We specifically use the two charts, R_t^L starting at 0 and R_t^U starting at the upper boundary. This choice results in coupling of the CUSUM charts (see Figure 2) where after the charts coalesce, they remain exactly the same. It is clear that after coupling, only one CUSUM chart needs to be considered. Coupling would not occur when using two CUSUM charts that are restarted - this feature is specifically for non-restarting CUSUM charts with an upper boundary.

A natural question to ask about coupling is: how long do the two non-restarting CUSUM charts take to couple? Denote the coupling time as $T = \min\{t \geq 1 : R_t^L = R_t^U\}$. Denote $\nu_\uparrow = \min\{t \geq 1 : R_t^L = h\}$ and $\nu_\downarrow = \min\{t \geq 1 : R_t^U = 0\}$ as the first time when the lower CUSUM chart reaches the upper boundary and the upper CUSUM chart reaches 0 respectively. As $T = \min(\nu_\uparrow, \nu_\downarrow)$ it follows that $E(T) \leq \min(E(\nu_\uparrow), E(\nu_\downarrow))$ for all $P \in \mathcal{P}$, since all CUSUM charts of the form (1) with any starting value, couple either at value 0 or the upper boundary. For $P \in \mathcal{P}$ such that $\mathcal{L}(X_t) = F_1$ for all $t \in \mathbb{N}$ we would expect

$$E(T) \leq E(\nu_\uparrow) \ll E(\nu_\downarrow)$$

and similarly for $P \in \mathcal{P}$ such that $\mathcal{L}(X_t) = F_0$ all $t \in \mathbb{N}$,

$$E(T) \leq E(\nu_\downarrow) \ll E(\nu_\uparrow).$$

Proving such coupling results could be a topic for further research.

Appendix 1

Proof of Theorem 1

We proceed to show the following: for any signal process Z and $j \in \{0, 1\}$

$$\mathbf{C}^j(Z) \geq \mathbf{D}^j(Z) \geq \mathbf{D}^j(Z^j) = \mathbf{C}^j(Z^*).$$

We start by showing $\mathbf{C}^1(Z) \geq \mathbf{D}^1(Z)$ for any signal process Z . As a first step we show that

$$\begin{aligned} \mathbf{C}^1(Z) \geq \\ \sup_{n \in \mathbb{N}} \max \left\{ \text{esssup } E \left([\tau_1^1(Z) - n + 1]^+ \mid \mathcal{F}_{n-1} \right) : P \in \mathcal{P}, \mathcal{L}(X_t) = F_1 \forall t \geq n \right\}. \end{aligned} \quad (5)$$

Consider the stopping time in $\mathbf{C}_n^1(Z)$, namely $\tau_n^1(Z) - n + 1$ and $[\tau_1^1(Z) - n + 1]^+$. For all $j \geq 1$ and fix $n \in \mathbb{N}$ we have

$$\{\tau_n^1(Z) - n + 1 = j\} = \{Z_m \neq 1 \text{ for } n \leq m < (j + n - 1), Z_{j+n-1} = 1\}$$

and

$$\{[\tau_1^1(Z) - n + 1]^+ = j\} = \{Z_m \neq 1 \text{ for } 1 \leq m < (j + n - 1), Z_{j+n-1} = 1\}.$$

Thus $\{[\tau_1^1(Z) - n + 1]^+ = j\} \subseteq \{\tau_n^1(Z) - n + 1 = j\}$ and so have a pointwise order of the stopping times. This implies (5).

Next we show that

$$\begin{aligned} \sup_{n \in \mathbb{N}} \max \left\{ \text{esssup } E \left([\tau_1^1(Z) - n + 1]^+ \mid \mathcal{F}_{n-1} \right) : P \in \mathcal{P}, \mathcal{L}(X_t) = F_1 \forall t \geq n \right\} \\ \geq \mathbf{D}^1(Z). \end{aligned} \quad (6)$$

For a fixed $n \in \mathbb{N}$

$$\begin{aligned} \max \left\{ \text{esssup } E \left([\tau_1^1(Z) - n + 1]^+ \mid \mathcal{F}_{n-1} \right) : P \in \mathcal{P}, \mathcal{L}(X_t) = F_1 \forall t \geq n \right\} \\ \geq \text{esssup } E \left([\tau_1^1(Z) - n + 1]^+ \mid \mathcal{F}_{n-1} \right) = \mathbf{D}_n^1(Z) \end{aligned} \quad (7)$$

where, on the right hand-side of the inequality, is under $P \in \mathcal{P}$ such that $\mathcal{L}(X_t) = F_0$ for all $t < n$ and $\mathcal{L}(X_t) = F_1$ for all $t \geq n$. Taking the supremum of (7) over $n \in \mathbb{N}$ shows (6). This completes showing $\mathbf{C}^1(Z) \geq \mathbf{D}^1(Z)$. A similar argument can be used to show $\mathbf{C}^0(Z) \geq \mathbf{D}^0(Z)$.

Since $B^j(Z) \geq c_j$ it follows that $B_1^j(Z) \geq c_j$ for $j \in \{0, 1\}$. As $B_1^j(Z)$ is the expectation in (2) it follows that (Ritov, 1990, Proposition 2) $\mathbf{D}^j(Z) \geq \mathbf{D}^j(Z^j)$ for any signal process Z .

A consequence of Moustakides (1986, Lemma 1) is that $\mathbf{D}^1(Z^1) = \mathbf{D}_1^1(Z^1)$. Thus, by definition of $\mathbf{D}_1^1(Z^1)$,

$$\mathbf{D}^1(Z^1) = E \left(\inf \{t \geq 1 : S_t^L \geq k_L\} \mid P \in \mathcal{P} : \mathcal{L}(X_t) = F_1 \forall t \in \mathbb{N} \right). \quad (8)$$

Replacing $\inf \{t \geq 1 : S_t^L \geq k_L\}$ in (8) with $\inf \{t \geq 1 : R_t^L \geq k_L\}$ as they are the same, gives

$$\mathbf{D}^1(Z^1) = \mathbf{C}_1^1(Z^*).$$

Thus, to show $\mathbf{D}^1(Z^1) = \mathbf{C}^1(Z^*)$, it suffices to show $\mathbf{C}^1(Z^*) = \mathbf{C}_n^1(Z^*)$ for all $n \in \mathbb{N}$.

To show this we follow an argument similar to that used in Moustakides (1986, Lemma 1). For any $m > n \geq 1$ and for fixed $\{X_{n+1}, \dots, X_m\}$, the quantity R_m^L is a non-decreasing function of R_n . This implies that the stopping time $\tau_n^1(Z^*) = \inf\{t \geq n : R_t^L \geq k_L\}$ is non-increasing with R_{n-1} . Thus the essential supremum in $\mathbf{C}_n^1(Z^*)$ is achieved for $R_{n-1} = 0$. Moreover, the essential supremum in $\mathbf{C}_n^1(Z^*)$ remains unchanged when taking the maximum over $P \in \mathcal{P}$ such that $\mathcal{L}(X_t) = F_1$ for $t \geq n$. Hence from stationarity all $\mathbf{C}_n^1(Z^*)$ are equal. This completes showing $\mathbf{D}^1(Z^1) = \mathbf{C}^1(Z^*)$.

We now show that $\mathbf{D}^0(Z^0) = \mathbf{D}^0(Z^*)$. Similarly, we have $\mathbf{D}^0(Z^0) = \mathbf{D}_1^0(Z^0)$ thus by definition of $\mathbf{D}_1^0(Z^0)$

$$\mathbf{D}^0(Z^0) = E(\inf\{t \geq 1 : S_t^U \geq k_U\} \mid P \in \mathcal{P} : \mathcal{L}(X_t) = F_0 \forall t \in \mathbb{N}). \quad (9)$$

Replacing $\inf\{t \geq 1 : S_t^U \geq k_U\}$ in (9) with $\inf\{t \geq 1 : R_t^U \leq h - k_U\}$ as they are the same, gives $\mathbf{D}^0(Z^0) = \mathbf{C}_1^0(Z^*)$. Showing that $\mathbf{C}^0(Z^*) = \mathbf{C}_n^0(Z^*)$ for all $n \in \mathbb{N}$ follows from the same argument given above. Thus $\mathbf{D}^0(Z^0) = \mathbf{C}^0(Z^*)$.

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